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ESTIMATION OF THE UNCERTAINTIES OF MULTI- PARAMETRIC INDIRECT MEASUREMENT SYSTEMS WITH FEW EXAMPLES

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Uncertainties of indirect multivariable measurement systems

Many physical quantities or parameters have to be measured indirectly, i.e. other measurands should be measured and from them the tested quantities (observables) should be determined. The set of such jointed variables is. the multivariate or vector measurand.
The flow chart of multi-parametric measurements is shown in Fig.1.

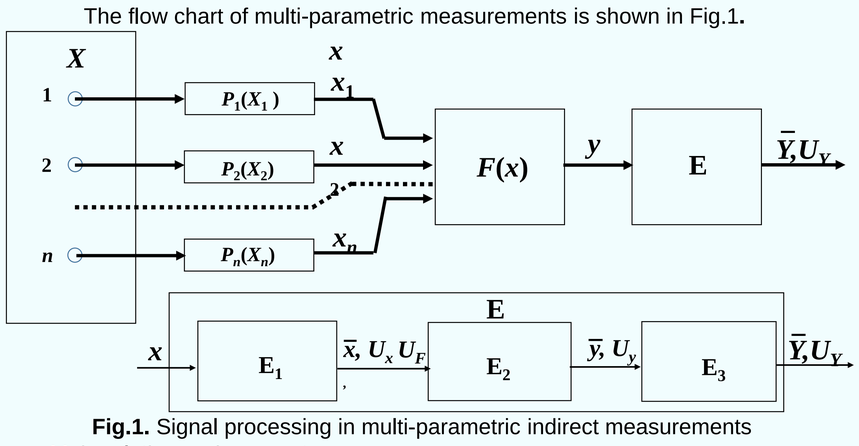

Fig.1. Signal processing in multi-parametric indirect measurements

Main relation and parameters:

$$y = F(x)$$

x= $[x_1, x_2, \ldots, x_n]$; $y = [y_1, y_2, \ldots, y_m]$ – vectors of input x_i and output y_i signals,
$\bar{x}$, U_x $U_{\delta X}$ – vector of estimators of $\bar{x}_i$ of input signals x and of its covariances matrixes of std. absolute and relative uncertainties,
$\bar{y}$, U_y, $U_{\delta Y}$ – estimators of mean values , covariaces matrixes of absolute and relative standard uncertainties of y,
$F(x)$, $U_{\delta F}$ – functional of processing x to y and the matrix U_F , $U_{\delta F}$ absolute and relative uncertainties of algorithm of functional F, E – processing unit for result of vector y,

Existing state

1. Description of accuracy of the instrumental measurement system for whole ranges of y_i by relative limited errors:

$$|\varepsilon_{y_i}|_{max} \le \left|\frac{\Delta_{y_{i0}}}{y_{i\,max}}\right| + \left|\varepsilon_{y_i}\frac{y_i}{y_i\,max}\right| \text{ and } i=1,\ldots,m,$$

where ε_{y_i}- relative errors for coponent y_i , $\Delta_{y_{i0}}$- aboslute errors of zero set values, $y_i\,max$- maxiumum measurement range for coponent y_i .

2. Method (GUM Supplement 2) of estimation standard uncertainties of single y_i values of vector Y components

$$U_Y = S \cdot U_X \cdot S^T,$$

Where: sensitivity matrix $S = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}$, covariance matrixes U_X, U_Y

In particular three dimensional case of U_X and U_Y are defined as:

$$U_X = \begin{bmatrix} \sigma_{x_1}^2 & \rho_{x_1x_2}\sigma_{x_1}\sigma_{x_2} & \rho_{x_1x_3}\sigma_{x_1}\sigma_{x_3} \\ \rho_{x_1x_2}\sigma_{x_1}\sigma_{x_2} & \sigma_{x_2}^2 & \rho_{x_2x_3}\sigma_{x_2}\sigma_{x_3} \\ \rho_{x_1x_3}\sigma_{x_1}\sigma_{x_3} & \rho_{x_2x_3}\sigma_{x_2}\sigma_{x_3} & \sigma_{x_3}^2 \end{bmatrix},$$

$$U_Y = \begin{bmatrix} \sigma_{y_1}^2 & \rho_{y_1y_2}\sigma_{y_1}\sigma_{y_2} & \rho_{y_2y_3}\sigma_{y_2}\sigma_{y_3} \\ \rho_{y_1y_2}\sigma_{y_1}\sigma_{y_2} & \sigma_{y_2}^2 & \rho_{y_2y_3}\sigma_{y_2}\sigma_{y_3} \\ \rho_{y_1y_3}\sigma_{y_1}\sigma_{y_3} & \rho_{y_2y_3}\sigma_{y_2}\sigma_{y_3} & \sigma_{y_3}^2 \end{bmatrix}.$$

What is needed

Description of the accuracy of vector instrumental systems by relative uncertainties (standard and expanded for given P probability of level of confidence).

If values of initial errors $|\Delta_{y_{i0}}| \ll |\varepsilon_{y_i} y_i|$, then for relative uncertainties covariance equation has analogical form as for absolute ones (we marked relative uncertainty as u_δ, so $u_\delta \equiv u_r$ according suplement 2 to the GUM).

$$U_{\delta Y} = S_\delta \cdot U_{\delta X} \cdot S_\delta^T$$

where : S_δ – matrix of performance function $F(x)$ linearized by first partial derivatives of input quantities, i.e.

$$S_\delta = \begin{bmatrix} \frac{x_1}{y_1}\frac{\partial y_1}{\partial x_1} & \cdots & \frac{x_n}{y_1}\frac{\partial y_1}{\partial x_n} \\ \cdots & \cdots & \cdots \\ \frac{x_1}{y_m}\frac{\partial y_m}{\partial x_1} & \cdots & \frac{x_n}{y_m}\frac{\partial y_m}{\partial x_n} \end{bmatrix}$$

and covariance matrices

$$U_{\delta X} = \begin{bmatrix} \delta_{x_1}^2 & \rho_{x_1x_2}\delta_{x_1}\delta_{x_2} & \rho_{x_2x_3}\delta_{x_2}\delta_{x_3} \\ \rho_{x_1x_2}\delta_{x_1}\delta_{x_2} & \delta_{x_2}^2 & \rho_{x_2x_3}\delta_{x_2}\delta_{x_3} \\ \rho_{x_2x_3}\delta_{x_2}\delta_{x_3} & \rho_{x_2x_3}\delta_{x_2}\delta_{x_3} & \delta_{x_3}^2 \end{bmatrix},$$

$$U_{\delta Y} = \begin{bmatrix} \delta_{y_1}^2 & \rho_{y_1y_2}\delta_{y_1}\delta_{y_2} & \rho_{y_2y_3}\delta_{y_2}\delta_{y_3} \\ \rho_{y_1y_2}\delta_{y_1}\delta_{y_2} & \delta_{y_2}^2 & \rho_{y_2y_3}\delta_{y_2}\delta_{y_3} \\ \rho_{y_1y_3}\delta_{y_1}\delta_{y_3} & \rho_{y_2y_3}\delta_{y_2}\delta_{y_3} & \delta_{y_3}^2 \end{bmatrix}$$

The estimation of vector $\bar{y}$ of results $\bar{y}_i$ and matrix of their relative uncertainties $U_{\delta Y}$ obtained by indirect measurements of m-dimensional signal y, depends on the measurement functional $F(x)$. All, or some components of the measurement result vector $\bar{y}$ of the measurand y can be used further or separately or jointly. In the latter case it is necessary to take in considerations the correlations between the uncertainties of components y_i of the output multidimensional measurand y.

The international recommendations for practical application of the method of determination of both estimators are described in Supplement 2 to the GUM [1], which does not cover situations, when the functional $F(x)$ is not accurate. For example, that is due to approximation of transfer functions, limited frequency ranges and signal processing by AC/DC converters, analogue multipliers, and other functional elements, necessary in instrumental measurement systems. Therefore $F(x)$ is also saddled with uncertainties U_F[1] Even in the most precise measurements the rounding of results also becomes essential, including the one resulting on the processing precision in digital circuits [2-4].

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[1] the point of view wider then metrology, i.e. **the measurement *science*** - the term proposed by prof. Ludvik Finkelstein from City University of London in 1970-s years during his activity in IMEKO.

Measurement of three resistances in Wheastone bridge circuit

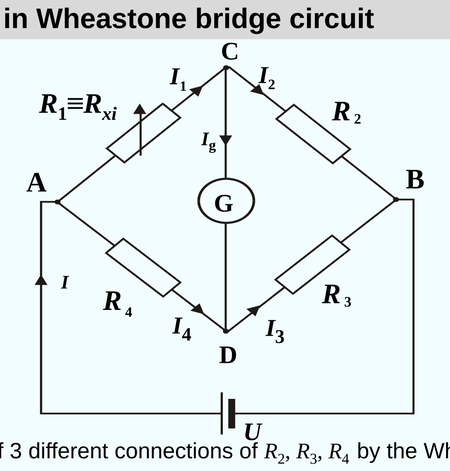


Fig 2. Measurement of 3 different connections of R_2, R_3, R_4 by the Wheastone bridge.

The measurements of three times by Wheastone bridge made from the balancing resistor R_1 and 3 tested resistors connected in different order ($R_2\,R_3\,R_4$; $R_2\,R_4\,R_3$ and $R_3\,R_2,\,R_4$) in the circuit loop. Three values R_{x1}, R_{x2}, R_{x3} of the balance resistance are

$$R_{x1} = R_2 \cdot R_4/R_3\,, \qquad R_{x2} = R_2 \cdot R_3/R_4, \qquad R_{x3} = R_3 \cdot R_4/R_2$$

From these relations the following values of tested resistances can be calculated

$$R_2 = \sqrt{R_{x1}\,R_{x2}}\,, \qquad R_3 = \sqrt{R_{x2}\,R_{x3}}\,, \qquad R_4 = \sqrt{R_{x1}\,R_{x3}}$$

The sensitivity matrix for relative uncertainties is:

$$S_\delta = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

For non-correlated input variables the matrix of output std. relative uncertainties is

$$U_{\delta y} = S_\delta \cdot \begin{bmatrix} \delta_{R_{x1}}^2 & 0 & 0 \\ 0 & \delta_{R_{x1}}^2 & 0 \\ 0 & 0 & \delta_{R_{x1}}^2 \end{bmatrix} \cdot S_\delta^T = \frac{1}{4}\begin{bmatrix} \delta_{R_{x1}}^2 + \delta_{R_{x2}}^2 & \delta_{R_{x2}}^2 & \delta_{R_{x1}}^2 \\ \delta_{R_{x2}}^2 & \delta_{R_{x2}}^2 + \delta_{R_{x3}}^2 & \delta_{R_{x3}}^2 \\ \delta_{R_{x1}}^2 & \delta_{R_{x3}}^2 & \delta_{R_{x1}}^2 + \delta_{R_{x3}}^2 \end{bmatrix}$$

So the standard relative output quantities are defined:

$$\delta_{R_2} = \frac{1}{2}\sqrt{\delta_{R_{x1}}^2 + \delta_{R_{x2}}^2}, \quad \delta_{R_3} = \frac{1}{2}\sqrt{\delta_{R_{x2}}^2 + \delta_{R_{x3}}^2}, \quad \delta_{R_3} = \frac{1}{2}\sqrt{\delta_{R_{x1}}^2 + \delta_{R_{x3}}^2},$$

and correlations coefficients:

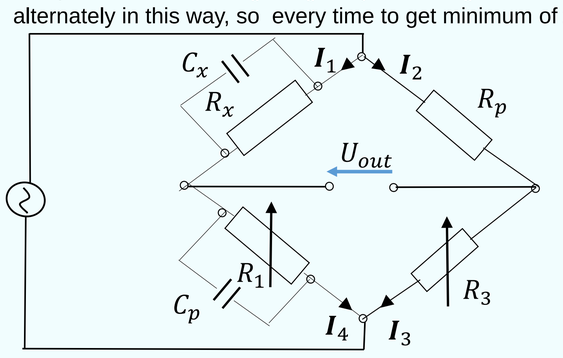
$$\rho_{R_2R_3} = \frac{1}{\sqrt{1+\left(\frac{\delta_{R_{x1}}}{\delta_{R_{x2}}}\right)^2}\sqrt{1+\left(\frac{\delta_{R_{x3}}}{\delta_{R_{x2}}}\right)^2}} > 0\,, \rho_{R_2R_4} = \frac{1}{\sqrt{1+\left(\frac{\delta_{R_{x2}}}{\delta_{R_{x1}}}\right)^2}\sqrt{1+\left(\frac{\delta_{R_{x3}}}{\delta_{R_{x1}}}\right)^2}} > 0, \rho_{R_2R_3} = \frac{1}{\sqrt{1+\left(\frac{\delta_{R_{x2}}}{\delta_{R_{x3}}}\right)^2}\sqrt{1+\left(\frac{\delta_{R_{x1}}}{\delta_{R_{x3}}}\right)^2}} > 0$$

Wien's bridge for measurement of two terminal RC in parallel structure

The balance condition of four armed AC bridge (supplied by the sinusoidal current source), as a result of $U_{out} = 0$ has the general complex form

$$Z_x Z_3 = Z_1 Z_2$$

The process of balancing bridge rely on setting of resistance R_1 and R_3, alternately in this way, so every time to get minimum of zero detection.



$$R_x = \frac{R_1 R_p}{R_3}$$

$$C_x = \frac{R_3 C_p}{R_p}$$

Fig 3. The Wien bridge for measurement the 2-terminal parallel structure of C_X // R_X

From the equality of real and imaginary components of complex **Z** we obtain formulas for measured output parameters R_x , C_x of two terminal element :

$$R_x = \frac{R_1 R_p}{R_3} \quad , \quad C_x = \frac{C_p R_3}{R_p}$$

Very often the dissipation coefficient D has to be determined additionally, as it characterizes the dissipation of energy in the real capacitor. For pulsation ω

$$D = \omega C_p R_1$$

If we assume that bridge parameters as input quantities are not correlated, then their covariance matrix of relative uncertainties has the form:

$$U_{\delta X} = \begin{bmatrix} \delta_{R_1}^2 & 0 & 0 & 0 & 0 \\ 0 & \delta_{R_3}^2 & 0 & 0 & 0 \\ 0 & 0 & \delta_{R_p}^2 & 0 & 0 \\ 0 & 0 & 0 & \delta_{C_p}^2 & 0 \\ 0 & 0 & 0 & 0 & \delta_\omega^2 \end{bmatrix}, \text{ matrix } S_\delta = \begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

and

$$U_{\delta Y} = S_\delta \cdot U_{\delta X} \cdot S_\delta^T = \begin{bmatrix} \delta_{R_1}^2 + \delta_{R_3}^2 + \delta_{R_p}^2 & -(\delta_{R_3}^2 + \delta_{C_p}^2) & -\delta_{R_1}^2 \\ -(\delta_{R_3}^2 + \delta_{C_p}^2) & \delta_{R_3}^2 + \delta_{R_p}^2 + \delta_{C_p}^2 & -\delta_{C_p}^2 \\ -\delta_{R_1}^2 & -\delta_{C_p}^2 & \delta_{R_1}^2 + \delta_{C_p}^2 + \delta_\omega^2 \end{bmatrix},$$

$$\delta_{R_x} = \sqrt{\delta_{R_1}^2 + \delta_{R_3}^2 + \delta_{R_p}^2}; \; \delta_{C_x} = \sqrt{\delta_{R_3}^2 + \delta_{R_p}^2 + \delta_{C_p}^2}\; ; \; \delta_D = \sqrt{\delta_{R_1}^2 + \delta_{C_p}^2 + \delta_\omega^2}$$

As we see from non-diagonal elements of $U_{\delta Y}$, pairs of relative uncertainties δ_{R_x} , δ_{C_x} , δ_{D_x} of output quantities are correlated with negative coefficients.

In particular case, when $\delta_{R_1} = \delta_{R_3} = \delta_{R_p} = \delta_{C_p} = \delta_{C_p} = \delta$, then

$$\delta_{R_x} = \delta_{C_x} = \delta_D = \sqrt{3}\,\delta, \quad \rho_{R_xC_x} = -\frac{2}{3}, \quad \rho_{R_xD} = \rho_{C_xD} = -\frac{1}{3}.$$

The parameter $w = 1 + 2\,\rho_{R_xC_x}\,\rho_{R_xD}\,\rho_{C_xD} - \rho_{R_xC_x}{}^2 - \rho_{R_xD}{}^2 - \rho_{C_xD}{}^2 = \frac{5}{27} > 0$

and the uncertainty coverage region of Gauss distribution has the 3D ellipsoidal form..

A star circuits

Let us condsider the case where three values of resistances are determined indirectly from three experimental results of measuring between the terminals A,B,C. Next this values coming from measurements are transferring to the module of performance F in which finally the values of resistance parameters of stars are calculated.

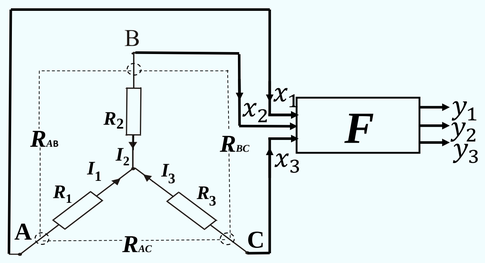


Fig. 4. The diagram of the star circuits with module of performance measurements

The measurement equations :are

$$R_{AC} = R_1 + R_3 \qquad R_{AB} = R_1 + R_2 \qquad R_{BC} = R_2 + R_3$$

It has been carried out by measurement of resistances R_{AB}, R_{BC}, R_{CA}

$$R_1 = \frac{R_{AB}}{2} - \frac{R_{BC}}{2} + \frac{R_{AC}}{2}; \quad R_2 = \frac{R_{AB}}{2} + \frac{R_{BC}}{2} - \frac{R_{AC}}{2}; \; R_3 = -\frac{R_{AB}}{2} + \frac{R_{BC}}{2} + \frac{R_{AC}}{2}$$

A corrections have been implemented in known systematic errors and unknowns errors have been randomized in uncertainty type B u_B. Next the resultants od absolute of standard uncertainties have been determined σ_{AB} , σ_{BC}, σ_{AC} (as a square of quadratic values of uncertainties type A u_A and B u_B) and also relative uncertainties are calculated δ_{AB} , δ_{BC}, δ_{AC}.

the Jacobian matrix e.g. sensitive matrix of coefficients is

$$S = \begin{bmatrix} \partial R_1/\partial R_{AB} & \partial R_1/\partial R_{BC} & \partial R_1/\partial R_{AC} \\ \partial R_2/\partial R_{AB} & \partial R_2/\partial R_{BC} & \partial R_2/\partial R_{AC} \\ \partial R_3/\partial R_{AB} & \partial R_3/\partial R_{BC} & \partial R_3/\partial R_{AC} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

We consider case when the absolute of uncertainties of input quantities are correlated σ_{AB} ,σ_{BC}, σ_{AC} and then in covariance matrix of input quantities appearing in general non zero elements in non diagonal positions defined by correlation coefficients ρ_{AB}, ρ_{BC}, ρ_{AC}

$$U_X = \begin{bmatrix} \sigma_{AB}^2 & \rho_{AB}\,\sigma_{AB}\,\sigma_{BC} & \rho_{BC}\sigma_{AB}\,\sigma_{AC} \\ \rho_{AB}\,\sigma_{AB}\,\sigma_{BC} & \sigma_{BC}^2 & \rho_{AC}\sigma_{BC}\sigma_{AC} \\ \rho_{BC}\sigma_{AB}\,\sigma_{AC} & \rho_{AC}\sigma_{BC}\sigma_{AC} & \sigma_{AC}^2 \end{bmatrix}$$

Summary of particular solutions

If $\sigma_{AB} = \sigma_{BC} = \sigma_{AC} = \sigma_{in}$, $\rho_{AB} = \rho_{BC} = \rho_{AC} = \rho_{in}$; $\sigma_{out} = \sigma_{y1} = \sigma_{y2} = \sigma_{y3} = \frac{1}{2}\sigma_{in}\sqrt{3 - 2\rho_{in}}$; $\rho_{y1y2} = \rho_{y1y3} = \rho_{y2y3} = \frac{\rho_{in}-1}{3-2\rho_{in}}$; $\rho_{y1y2}, \rho_{y2y3}, \rho_{y1y3} \le 0$.

if $\rho_{in} = 0$;then $\sigma_{out} = \frac{\sqrt{3}}{2}\sigma_{in}$ and $\rho_{y1y2} = \rho_{y1y3} = \rho_{y2y3} = -\frac{1}{3}$; half axises; $1{,}4\sigma_{in}$,$2{,}8\,\sigma_{in}$,$2{,}8\,\sigma_{in}$

if $\rho_{in} = 1$ then min $\sigma_{out} = \frac{1}{2}\sigma_{in}$, $\rho_{y1y2} = \rho_{y1y3} = \rho_{y2y3} = 0$; radius $1{,}4\,\sigma_{in}$

if $\rho_{in} = -1$ then max $\sigma_{out} = \frac{\sqrt{5}}{2}\sigma_{in}$, $\rho_{y1y2} = \rho_{y1y3} = \rho_{y2y3} = -\frac{2}{5}$; half axises $1{,}4\sigma_{in}$,$3{,}7\,\sigma_{in}$,$3{,}7\,\sigma_{in}$

Ellipsoidal coverage region with probability 0,95 in this special case for relative uncertainties determines the ellipsoid with half axis a=1,449δ, b=6,261δ, c=5,409δ, in formulas it is used that the coverage factor/ extension/ coefficient for 95% of coverage region in three dimensional GAUSS distribution equal k_p=2,8. The ellipsoid is contiguous in six points to the cube with edges equal $2 \cdot 2{,}8\sqrt{3}\,\delta = 9{,}7\,\delta$ containing ellipsoid.

Covariance matrix and cover region for three dimensional case

The coverage region of three dimensional uncertainty for Gaussian/t-Student distribution is created by ellipsoid (Fig. 2) for correlation coefficients satisfy the condition

$$\mathbf{1} > \; w = 1 + 2\,\rho_{y_1y_2} \cdot \rho_{y_2y_3} \cdot \rho_{y_2y_3} - \rho_{y_1y_2}{}^2 - \rho_{y_1y_3}{}^2 - \rho_{y_2y_3}{}^2 > \mathbf{0}$$

Then characteristic equations for three dimensional inverse of covariant matrix is[2]:

$$\lambda^3 - \varepsilon \cdot \lambda^2 + \cdot K \cdot \lambda - L = 0$$

Above eequation of third order has three positive real roots. The half axes of the ellipsoidal are described by Cardano formulas, as follows:

$$a^2 = \frac{k_p^2}{\lambda_1} = \frac{\gamma}{\cos\frac{\psi}{3}}\,, \; b^2 = \frac{k_p^2}{\lambda_2} = \frac{\gamma}{\cos\frac{\psi + 2\cdot\pi}{3}}\,, \quad c^2 = \frac{k_p^2}{\lambda_3} = \frac{\gamma}{\cos\frac{\psi + 4\cdot\pi}{3}}.$$

where k_p - expanded coefficient of uncertainty; coefficient γ and angle $0 < \psi < \pi$ are defined as:

$$\gamma = \frac{3k_p^2}{\varepsilon + 2\sqrt{\varepsilon^2 - 3K}}\,, \qquad \cos\psi = \frac{27 \cdot L + 2 \cdot \varepsilon^3 - 9 \cdot \varepsilon \cdot K}{2\sqrt{(\varepsilon^2 - 3 \cdot K)^3}}.$$

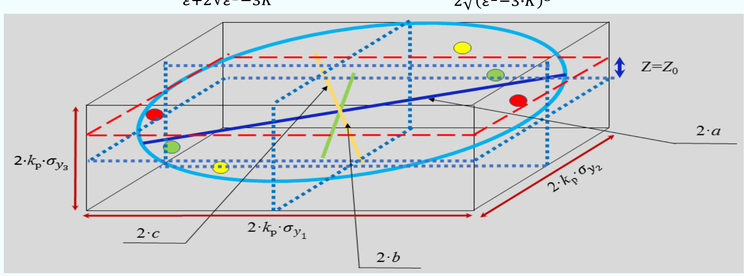


Fig. 5. Three dimensional cover region – ellipsoid region

Summary and Conclusion

Few examples of determining the relative uncertainties of multi-parameter linear and nonlinear formulas have been presented. Considered are: indirect measurements of three resistances due their three different connections as arms of Wheatstone bridge, measurement of paralel RC circuit by Wien bridge and measurement of resistances of the star circuit from its terminals.

Covariance matrix of relative uncertainties is introduced as proper tool applied for describing the accuracy of whole ranges of the multidimensional indirect measurement systems.

Literature

[1] JCGM 102:2011, Evaluation of measurement data – Supplement 2 to the Guide to the expression of uncertainty in measurement"– Extension to any number of output quantities.
[2] Warsza, Z.L.: *Evaluation and Numerical Presentation of the Results of Indirect Multivariate Measurements. Outline* …. In: Advanced Mathematical & Computational Tools in Metrology and Testing IX, ed. by Franco Pavese et all, Series on Advances in Mathematics for Applied Sciences, vol. 84, World Scientific 2012, Singapore p.418-425
[3] Warsza, Z.L.: *About evaluation of multivariate measurements results.* JAMRIS (Journal of Automation, Mobile Robotics & Intelligent Systems, ed. by PIAP) vol.6, n.4 2012, p. 27-32
[4] Harris, P. M., Cox, M. G. (2014). On a Monte Carlo method for measurement uncertainty evaluationand its implementation. *Metrologia*, 51, S176–S182.
[5] Willink, R., Hall, B. D. (2002). A classical method for uncertainty analysis with multidimensional data. *Metrologia*, 39, 361–369.
[6] Hall, B. D. (2003). Calculating measurement uncertainty for complex-valued quantities. *Measurement Science and Technology*, 14, 368–375.
[7] Wang, C. M., Iyer H. K. (2006). Uncertainty analysis for vector measurands using fiducial inference. *Metrologia*, *43*, 486–494.
[8] Hasselbarth, W., Bremser, W. (2007). Measurement uncertainty for multiple measurands: characterization and comparison of uncertainty matrices. *Metrologia*, 44, 128–145. 17
[9] Hall, B. D. (2013). Expanded uncertainty regions for complex quantities. *Metrologia*, 50, 490–498.
[10] Ramos, P. M., Janeiro, F. M., Girao, P. S. (2016). *Uncertainty evaluation of multivariate quantities: A case study on electrical impedance.* Measurement, 78, 397–411.
[11] Hall, B. D. (2016). Evaluating the measurement uncertainty of complex quantities: a selective review. *Metrologia*, 53, S25–S31.
[12] Finkelstein L. ACTA IMEKO May 2014, Volume 3, Number 1, 10 – 15 www.imeko.org_May 2014 | Volume 3 | Number 1 | 10 Fundamental concepts of measurement.

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[2]. where coefficients: $\varepsilon = \frac{1}{w}\left(\frac{1-\rho_{y_2y_3}{}^2}{\delta_{y_1}^2} + \frac{1-\rho_{y_1y_3}{}^2}{\delta_{y_2}^2} + \frac{1-\rho_{y_1y_2}{}^2}{\delta_{y_3}^2}\right)$, $K = \frac{1}{w}\left(\frac{1}{\delta_{y_1}^2\delta_{y_2}^2} + \frac{1}{\delta_{y_2}^2\delta_{y_3}^2} + \frac{1}{\delta_{y_1}^2\delta_{y_3}^2}\right)$ and $L = \frac{1}{w\delta_{y_1}^2\delta_{y_2}^2\delta_{y_3}^2}$